Back-up Exam ISI Bangalore January 2016

M.Math Linear Algebra 100 Points

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

(d) All vector spaces are assumed to be finite dimensional, unless mentioned otherwise.

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1. [18 points] Let $T\colon V \to V$ be a linear map of vector spaces and let $W \subset V$ be a T-invariant subspace.
   (i) Describe how T induces a natural linear map $\overline{T}\colon V/W \to V/W$.
   (ii) Prove that if $T\big|_W$ and $\overline{T}$ upper-triangulable, then so is T.

2. [14 points] Let $V = V_1 \oplus V_2$. Let W be a subspace of V.
   (i) If W contains V_1 prove that $W = V_1 \oplus (W \cap V_2)$.
   (ii) Prove or disprove: $W = (W \cap V_1) \oplus (W \cap V_2)$.

3. [12 points] Let $V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$ be an exact sequence of linear maps of vector spaces. Prove that $\operatorname{rank}(S) + \operatorname{rank}(T) = \dim(V_2)$.

5. [16 points] Let $(V, \langle\ ,\ \rangle)$ be Hermitian space. Prove that if $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^n$ are both a basis of V such that $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle$ for all i, j, then the unique operator $T\colon V \to V$ satisfying $T(v_i) = w_i$ is unitary.

6. [16 points] Classify upto similarity, all 6×6 matrices over $\mathbb{C}$ whose minimal polynomial is given by $p(t) = (t+1)(t-1)^3$.

7. [12 points] Let F be a field and let $a_0, \ldots, a_{n-1} \in F$. Give an example of an $n \times n$ matrix A over F whose characteristic polynomial is $t^n + a_{n-1}t^{n-1} + \cdots + a_0$.

8. [12 points] Prove that a real symmetric matrix which is nilpotent is the zero matrix.